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Extremal graphs with given order and the rupture degree[☆]

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ABSTRACT

The rupture degree of an incomplete connected graph G is defined by $r(G) = \max\{\omega(G - X) - |X| - \tau(G - X) : X \subset V(G), \omega(G - X) > 1\}$, where $\omega(G - X)$ is the number of components of $G - X$ and $\tau(G - X)$ is the order of a largest component of $G - X$. In Li and Li [5] and Li et al. (2005) [4], it was shown that the rupture degree can be well used to measure the vulnerability of networks. In this paper, the maximum and minimum networks with prescribed order and the rupture degree are obtained. Finally, we determine the maximum rupture degree graphs with given order and size.

1. Introduction

The stability of a communication network, composed of processing nodes and communication links, is of prime importance to network designers. As the network begins losing links or nodes, eventually there is a loss in its effectiveness. Thus, communication networks must be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network. Many graph theoretical parameters have been used in measuring the vulnerability of networks, such as connectivity, toughness [1], integrity [2], tenacity [3] and the rupture degree [4]. The rupture degree (also called the additive dual of the tenacity) is the newest and well used parameter, for it represents a trade-off between the amount of work done to damage the network and how badly the network is damaged, see [5,4].

Before we formally define the rupture degree of a graph, we recall some terminologies and notation from [4]. Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. A set $X \subset V(G)$ is a cut set of G, if either $G - X$ is disconnected or $G - X$ has only one vertex. For any cut set X of G, let $\omega(G - X)$ and $\tau(G - X)$ denote the number of components and the order of a largest component in $G - X$, respectively.

The rupture degree of G is defined as

$$r(G) = \max\{\omega(G - X) - |X| - \tau(G - X) : X \subset V(G), \omega(G - X) > 1\}.$$

In particular, the rupture degree of a complete graph K_n is defined to be $1 - n$.

A vertex cut set X of graph G is called an *r-set* of G if $r(G) = \omega(G - X) - |X| - \tau(G - X)$. A network is called *the maximum (minimum) network* if it has maximum (minimum) number of edges with prescribed order and some properties.

In this paper, we mainly consider the construction of the extreme graphs with given number of vertices and rupture degree. In Section 2, the maximum networks with prescribed order and the rupture degree are obtained. In Section 3, the minimum networks will be discussed. In Section 4, the maximum rupture degree graphs are characterized with prescribed order and size.

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Throughout this paper, all graphs are finite, undirected and simple. We use Bondy and Murty [6] for terminologies and notation not defined here. Let K_n, E_n and $G[X]$ denote complete graph, the null graph with n vertices and the induced subgraph of G on a nonempty subset $X \subset V(G)$, respectively. A comet $C_{t,s}$ is defined as the graph obtained by identifying one end of the path P_t $(t \geq 2)$ with the center of the star $K_{1,s}$ and the center is also called the center of the comet. The *Join* of two disjoint graphs G and H, denoted by $G + H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv | u \in V(G), v \in V(H)\}$.

2. Maximum graphs with given number of vertices and rupture degree

In this section, we characterize the graphs with maximum number of edges and given number of vertices and rupture degree. By the definition of rupture degree, if a graph G has n vertices and rupture degree $1-n$, then $G = K_n$. In the following we only consider graphs with n vertices and rupture degree $r \neq 1 - n$. First we give two useful lemmas which were proved in [4].

Lemma 2.1. *Let G be an incomplete connected graph with order n. Then $3 - n \leq r(G) \leq n - 3$.*

Lemma 2.2. *There exists no graph G with n vertices such that $r(G) = n - 4$. For any integer r with $3 - n \leq r \leq n - 5$ or $r = n - 3$, there exist graphs with n vertices and rupture degree r.*

Theorem 2.1. *Let n be an integer greater than 2, and r be an integer with $3 - n \leq r \leq n - 5$ or $r = n - 3$. If G is a graph which has the maximum number of edges among all graphs of order n and rupture degree r, then*

$$G = \begin{cases} K_{\frac{n-r-1}{2}} + \dfrac{n+r+1}{2} K_1, & \text{if } n + r \text{ is odd;} \\ K_{\frac{n-r-4}{2}} + \left(\dfrac{n+r-4}{2} K_1 \cup 2K_2\right), & \text{if } n + r \text{ is even.} \end{cases}$$

Proof. Let X be an *r-set* of G, i.e., $\omega(G-X) - |X| - \tau(G-X) = r(G)$. Suppose that the components of $G-X$ are $G_1, G_2, \ldots, G_k$, and let $|X| = x$, $|V(G_i)| = n_i$ for $i = 1, 2, \ldots, k$. Then $G - X = G_1 \cup G_2 \cup \cdots \cup G_k$ and $r(G) = k - x - \tau(G-X)$, $\sum_{i=1}^{k} n_i = n - x$ and $1 \leq n_i \leq n - x - (k-1) = n + 1 - x - k$ for $i = 1, 2, \ldots, k$.

Since G has the maximum number of edges among all graphs with n vertices and rupture degree r, we have

(1) $G[X]$ is a complete subgraph of G;
(2) All G_i $(i = 1, 2, \ldots, k)$ are complete subgraphs of G;
(3) All vertices in X are adjacent to all vertices in G_i $(i = 1, 2, \ldots, k)$.

Then $G = K_x + \left(\bigcup_{i=1}^{k} K_{n_i}\right)$. So we have

$$\begin{aligned} |E(G)| &= f(x, n_1, n_2, \ldots, n_k) \\ &= \binom{x}{2} + \sum_{i=1}^{k} \binom{n_i}{2} + x \sum_{i=1}^{k} n_i \\ &= \binom{x}{2} + \frac{1}{2}\left(\sum_{i=1}^{k} n_i\right)^2 + \left(x - \frac{1}{2}\right) \sum_{i=1}^{k} n_i - \sum_{1 \leq i < j \leq k} n_i n_j \\ &= \binom{x}{2} + \frac{1}{2}(n-x)^2 + \left(x - \frac{1}{2}\right)(n-x) - \sum_{1 \leq i < j \leq k} n_i n_j. \end{aligned}$$

Case 1. $r + n$ is odd.

First, let us determine the maximum value of $f(x, n_1, n_2, \ldots, n_k)$ for a given x by solving the following nonlinear integer programming

$$\min g(n_1, n_2, \ldots, n_k) = \sum_{1 \leq i < j \leq k} n_i n_j$$

$$\text{s.t.} \begin{cases} 1 \leq n_i \leq n - x - (k-1) & \text{for } i = 1, 2, \ldots, k \\ \displaystyle\sum_{i=1}^{k} n_i = n - x \\ n_i \in Z. \end{cases}$$

We set $N = (n_1, n_2, \ldots, n_k)$ for convenience. Let $N^0 = (n_1^0, n_2^0, \ldots, n_k^0)$ be an arbitrary feasible solution of the above nonlinear integer programming and suppose that n_j^0 is the first number larger than 1 among $n_1^0, n_2^0, \ldots, n_k^0$. Then construct a new feasible solution $N^1 = (\underbrace{1, 1, \ldots, 1}_{j}, n^0_{j+1} + n^0_j - 1, n^0_{j+2}, \ldots, n^0_k)$. It is clear that $g(N^1) \le g(N^0)$. Repeating the above process, we can finally get a feasible solution $N^* = (\underbrace{1, 1, \ldots, 1}_{k-1}, n - x - k + 1)$. Since N^0 is an arbitrary feasible solution, N^* is an optimal solution of the above nonlinear integer programming. Then the maximum value of $f(x, n_1, n_2, \ldots, n_k)$ for the given x is

$$p(x) = f(x, 1, 1, \ldots, 1, n - x - k + 1) = \binom{n - x - k + 1}{2} + \binom{x}{2} + x(n - x).$$

Next let us determine the maximum value of $p(x)$. By the above analysis, $\tau(G - X) = n_k = n - x - k + 1$. Thus $r = k - x - \tau(G - X) = 2k - n - 1$. This implies that $k = \frac{n+r+1}{2}$. Then by $\tau(G - X) \ge 1$ we get $x \le n - k = \frac{n-r-1}{2}$. Furthermore, since $p(x)$ is an increasing function when $1 \le x \le \frac{n-r-1}{2}$, then, $p(x)$ will meet the maximum value when $x = \frac{n-r-1}{2}$. Therefore, the maximum graph $G = K_{\frac{n-r-1}{2}} + \frac{n+r+1}{2} K_1$.

Case 2. $r + n$ is even.

Clearly, if $\tau(G-X) = 1$, then $n+r = n+k-x-\tau(G-X) = 2(n-x)-1$ is odd, a contradiction. If $\tau(G-X) = n-x-k+1$, then $n+r = 2k-1$ is odd, again a contradiction. So in this case $2 \le \tau(G-X) \le n-x-k$. In the following let us determine the maximum value of $f(x, n_1, n_2, \ldots, n_k)$ for a given x by solving the following nonlinear integer programming

$$\min g(n_1, n_2, \ldots, n_k) = \sum_{1 \le i < j \le k} n_i n_j$$

$$\text{s.t.} \begin{cases} 1 \le n_i \le n - x - k \quad \text{for } i = 1, 2, \ldots, k \\ \displaystyle\sum_{i=1}^{k} n_i = n - x \\ 2 \le \displaystyle\max_{1 \le i \le k} n_i \le n - x - k \\ n_i \in Z. \end{cases}$$

Similar to case 1, we can obtain that the optimal solution of the above nonlinear integer programming is $N^{**} = (\underbrace{1, 1, \ldots, 1}_{k-2}, 2, n - x - k)$. Thus the maximum value of $f(x, n_1, n_2, \ldots, n_k)$ for the given x is

$$q(x) = f(x, 1, 1, \ldots, 2, n - x - k) = \binom{n - x - k}{2} + \binom{x}{2} + x(n - x) + 1.$$

Clearly, $\tau(G - X) = n - x - k$, and then $r = k - x - \tau(G - X) = 2k - n$, this means $k = \frac{n+r}{2}$. At the same time, by $\tau(G - X) \ge 2$ get $x \le \frac{n-r-4}{2}$. Furthermore, since $q(x)$ is an increasing function when $1 \le x \le \frac{n-r-4}{2}$, then, $q(x)$ will meet the maximum value when $x = \frac{n-r-4}{2}$. Therefore, the maximum graph $G = K_{\frac{n-r-4}{2}} + \left(\frac{n+r-4}{2} K_1 \cup 2K_2\right)$.

The proof is completed. □

Corollary 2.1. *Let n be an integer greater than 2, and r an integer with $3 - n \le r \le n - 5$ or $r = n - 3$. If G is a graph which has the maximum number of edges among all graphs of order n and rupture degree r, then*

$$|E(G)| = \begin{cases} \dfrac{1}{8}(3n^2 - r^2 - 2nr - 4n + 1), & \text{if } n + r \text{ is odd;} \\ \dfrac{1}{8}(3n^2 - r^2 - 2nr - 6r - 10n + 8), & \text{if } n + r \text{ is even.} \end{cases}$$

3. Related discussion for minimum graphs with given number of vertices and rupture degree

In this section, we consider the minimum network with given order and rupture degree. Clearly, if $|V(G)| = n$ and $r(G) = 1 - n$, then $G = K_n$ and thus $\min |E(G)| = \frac{n(n-1)}{2}$. If $|V(G)| = n$ and $r(G) = n - 3$, then $G = K_{1,n-1}$ and $\min |E(G)| = n - 1$. As for the general cases, it seems complicated. In the following we discuss some special cases. We denote $T(n, \Delta)$ as the family of trees with order $n(\ge 3)$ and maximum degree $\Delta(\ge 2)$.

Lemma 3.1. *The rupture degree of the comet is $r(C_{t,s}) = \begin{cases} s - 1, & \text{if } t \text{ is even,} \\ s - 2, & \text{if } t \text{ is odd.} \end{cases}$*

Lemma 3.2. *Let T be a tree and X be an r-set of T, then $\tau(T - X) \le 2$.*

Proof. Assume that $\tau(T-X) > 2$. Without loss of generality, let $C_1, C_2, \ldots, C_k$ be all the largest components of $T-X$. It is clear that every C_j is a subtree of T and $|V(C_1)| = |V(C_2)| = \cdots = |V(C_k)| > 2$. Then select one vertex u_j in every $V(C_j)$ such that $d_{C_j}(u_j) \geq 2$, and let $X_1 = X \cup \{u_1, u_2, \ldots, u_k\}$. Clearly, $\omega(T-X) - |X| \leq \omega(T-X_1) - |X_1|$ and $\tau(T-X_1) \leq \tau(T-X) - 1$. Thus $\omega(T-X) - |X| - \tau(T-X) < \omega(T-X_1) - |X_1| - \tau(T-X_1)$. This is a contradiction to X is an r-set of T. Therefore $\tau(T-X) \leq 2$. □

Lemma 3.3. *Let T be a tree with maximum degree Δ, X be the r-set of T, then $\omega(T-X) - |X| \geq \Delta - 1$.*

Proof. Let X be an r-set of T. If $|X| = 1$, assume that $X = \{u\}$. By the definition of r-set we know that u must be the maximum degree vertex of T. Thus $\omega(T-X) - |X| = \Delta - 1$. If $|X| > 1$, by the definition of rupture degree there exists a maximum degree vertex $v \in X$. Combining this with the fact T is a tree we have that $\omega(T-X) - |X|$ is increasing with $|X|$. Thus $\omega(T-X) - |X| \geq \omega(T - \{v\}) - |\{v\}| = \Delta - 1$. The proof is completed. □

Theorem 3.1.

$$\min_{T \in T[n,\Delta]} r(T) = \begin{cases} \Delta - 2, & \text{if } n = \Delta + 1 \\ \Delta - 3, & \text{if } n > \Delta + 1. \end{cases}$$

Proof. Clearly, if $n = \Delta + 1$, T is unique and $T = K_{1,n-1}$, thus $r(T) = n - 3 = \Delta - 2$. In the following we consider the case $n > \Delta + 1$.

If $n = \Delta + 2$, T is also unique and $T = K_{1,n-1}^{+}$ obtained by subdividing one edge of the star $K_{1,n-1}$, thus $r(T) = n - 5 = \Delta - 3$. If $n \geq \Delta + 3$. By Lemmas 3.2 and 3.3 we easily have $r(T) \geq \Delta - 3$.

In fact, the results can be achieved by trees T_1 and T_2; while $n - \Delta$ is even, $T_1 = C_{n-\Delta+1,\Delta-1}$ and while $n - \Delta$ is odd, T_2 obtained by identifying one end of path P_3 with the center of the comet $C_{n-\Delta,\Delta-2}$. □

By Theorem 3.1, we immediately get the following corollary.

Corollary 3.1. *If $|V(G)| = n$ and $-1 \leq r(G) \leq n - 5$, then there exists a tree with order n and rupture degree r and thus* $\min |E(G)| = n - 1$.

As for the case $r(G) \leq -2$, it seems complicated to describe the minimum graph, maybe this problem needs some other parameters to co-consider. But the following observation is clear.

The case $r = -2$ ($n \geq 5$): If n is odd, the odd cycle C_n is the minimum graph. If n is an even, the graph G obtained by connecting one vertex of an odd cycle with one vertex of another odd cycle maybe the minimum graph. Thus we claim that $\min |E(G)| = 2\left\lfloor \frac{n}{2} \right\rfloor + 1$ when $r = -2$.

The case $r = -3$ ($n \geq 6$): The graph G obtained by adding edges u_1u_3, u_3u_5, u_2u_4 in cycle $C = u_1u_2u_3 \cdots u_nu_1$ may be the minimum graph. Thus we claim that $\min |E(G)| = n + 3$ when $r = -3$.

4. Graphs with maximum rupture degree and given order and size

In this section, we discuss the maximum rupture degree of graphs with prescribed order and size. Let G be a connected graph with order n and size m, then $n - 1 \leq m \leq \frac{1}{2}n(n-1)$. A clique of a simple graph G is a subset S of V such that the induced subgraph $G[S]$ is complete.

Theorem 4.1. *Let G be a connected graph among all graphs with order n and size m. If $\binom{k}{2} + (n-k)(k-1) < m \leq \binom{k}{2} + (n-k)k$, then the maximum rupture degree of G is $r(G) = n - 2k - 1$.*

Proof. Let G be a connected graph on n vertices and m edges with the maximum rupture degree. Denote by Π the family of all r-sets X in G. Let X^* be an element of Π with maximum order and $G_1, G_2, \ldots, G_p$ be all components of $G - X^*$. Next we prove that $\tau(G - X^*) = 1$.

Suppose that there are q components $G_1, G_2, \ldots, G_q$ such that $|G_i| \geq 2$ in $G - X^*$. Clearly, $0 \leq q \leq p$. In the following we distinguish two cases to prove that $q = 0$.

Case 1. Suppose that $q \geq 2$. Since G is connected, we can select a vertex $u_i \in V(G_i)$ for $i = 1, 2, \ldots, q$ such that u_i is adjacent to some vertex of X^*. Thus construct a new graph G' from graph G as $G' = G - \cup_{i=1}^{q-1} E_i' + \cup_{i=1}^{q-1} E_i''$, where $E_i' = \{u_iv_i^j | v_i^j \in N(u_i) \cap V(G_i)\}$, $E_i'' = \{u_qv_i^j | v_i^j \in N(u_i) \cap V(G_i)\}$. Clearly, G' is a connected graph on n vertices and m edges. Let $X' = X^* \cup \{u_q\}$, then X' is a cut set of graph G' and $\omega(G' - X') \geq \omega(G - X^*) + q - 1 \geq \omega(G - X^*) + 1$, $|X'| = |X^*| + 1$, $\tau(G' - X') \leq \tau(G - X^*) - 1$. Thus

$$\begin{aligned} r(G') &\geq \omega(G' - X') - |X'| - \tau(G' - X') \\ &\geq \omega(G - X^*) + 1 - |X^*| - 1 - \tau(G - X^*) + 1 \\ &= \omega(G - X^*) - |X^*| - \tau(G - X^*) + 1 \\ &= r(G) + 1 > r(G). \end{aligned}$$

But this contradicts the fact that G is a graph with the maximum rupture degree.

Case 2. Suppose that $q = 1$. Without loss of generality, assume that $|V(G_1)| \geq 2$. Now distinguish two cases to complete the proof.

Subcase 2.1. If $V(G_1)$ is a clique of G, for any vertex $u_1 \in V(G_1)$, let $X' = X^* \cup V(G_1)/\{u_1\}$, then $\omega(G - X') = \omega(G - X^*)$, $|X'| = |X^*| + \tau(G - X^*) - 1$, $\tau(G - X') = 1$. Clearly, $\omega(G - X') - |X'| - \tau(G - X') = \omega(G - X^*) - |X^*| - \tau(G - X^*)$. Hence, X' is an r-set with $|X'| > |X^*|$, which is a contradiction to the choice of X^*.

Subcase 2.2. If $V(G_1)$ is not a clique of G, it is clear that there exist a cut set X_0 of G_1, we let $X' = X^* \cup X_0$, then $\omega(G - X') \geq \omega(G - X^*) + 1$, $|X'| = |X^*| + |X_0|$, $\tau(G - X') \leq \tau(G - X^*) - |X_0| - 1$. Thus

$$\begin{aligned} \omega(G - X') - |X'| - \tau(G - X') &\geq \omega(G - X^*) + 1 - |X^*| - |X_0| - \tau(G - X^*) + |X_0| + 1 \\ &= \omega(G - X^*) - |X^*| - \tau(G - X^*) + 2 \\ &= r(G) + 2 > r(G). \end{aligned}$$

We thus have a contradiction to the definition of rupture degree of G.

By the above analysis, we have that $\tau(G - X^*) = 1$ for the r-set X^*. Denote $|X^*| = x$. Then

$$\begin{aligned} r(G) &= \omega(G - X^*) - |X^*| - \tau(G - X^*) \\ &= n - 2x - 1. \end{aligned}$$

In the following we prove: If $|E(G)| = m$ satisfies $\binom{k}{2} + (n-k)(k-1) < m \leq \binom{k}{2} + (n-k)k$ for a positive integer k, then the rupture degree $r(G) = n - 2x - 1$ of G will meet the maximum value while $x = k$.

First, it is clear that $x \geq k$. In fact, if $x \leq k - 1$, then

$$\begin{aligned} |E(G)| &\leq \binom{x}{2} + (n - x)x \\ &\leq \binom{k-1}{2} + (n - k + 1)(k - 1) \\ &= \binom{k}{2} + (n - k)(k - 1) \end{aligned}$$

which contradicts the value range of the number of edges of G. Thus we consider the case $x \geq k$ and easily get that $r(G) \leq n - 2k - 1$. At the same time, we let $G_0 = K_k + (n - k)K_1$, since $|E(G_0)| = \binom{k}{2} + (n - k)k$ and $r(G_0) = n - 2k - 1$, then $r(G) \geq r(G_0) = n - 2k - 1$. Hence, $r(G) = n - 2k - 1$.

The proof is completed. $\square$

Corollary 4.1. *Let n be an integer greater than 4, and m be an integer with $\binom{k}{2} + (n-k)(k-1) < m \leq \binom{k}{2} + (n-k)k$ for an integer k. Then the maximum rupture degree graph among all graphs with order n and size m is*

$$G = [K_k + (n - k)K_1] \setminus \{e_1, e_2, \ldots, e_t\}$$

where $t = \binom{k}{2} + k(n - k) - m$ and $e_1, e_2, \ldots, e_t$ are edges whose ends are respectively in $V(K_k)$ and $V(G \setminus K_k)$ and let G be connected.

5. Conclusions

In this paper we discuss some extremal properties related to rupture degree. The maximum and minimum networks with prescribed order and rupture degree are obtained, and determine the maximum rupture degree graphs with given order and size. As for the problem of determining the minimum networks and the minimum rupture degree graphs is much more complicated than that of the maximum case. Here we only discuss the minimum networks for $-1 \leq r(G) \leq n - 5$. In fact, the results of the minimum networks are more interesting. And thus it needs further research in this aspect.

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